

Laser Pulses as Measurements. Application to the Quantum Zeno Effect

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Abstract

Short pulses of a probe laser have been used in the past to measure whether a two-level atom is in its ground or excited state. The probe pulse couples the ground state to a third, auxiliary, level of the atom. Occurrence or absence of resonance fluorescence were taken to mean that the atom was found in its ground or excited state, respectively. In this paper we investigate to what extent this procedure results in an effective measurement to which the projection postulate can be applied, at least approximately. We discuss in detail the complications arising from an additional time development of the two-level system proper during a probe pulse. We extend our previous results for weak probe pulses to the general case and show that one can model an ideal (projection-postulate) measurement much better with a strong than a weak probe pulse. In an application to the quantum Zeno effect we calculate the slow-down of the atomic time development under n repeated probe pulse measurements and determine the corrections compared to the case of n ideal measurements.

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1. Introduction

For a quantum system a measurement in general involves a macroscopic apparatus which interacts with the system and leaves a pointer in some definite position. By an *ideal* measurement at time t we mean a measurement whose effect on a state can be described by applying the projection postulate at time t . The projection postulate as commonly used nowadays is due to Lüders [1]. For observables with degenerate eigenvalues his formulation differs from that of von Neumann [2]. The projection postulate has been widely regarded as a useful tool⁴. It is also known that one can envisage more general measurements where the projection postulate is not applicable; cf. e.g. [3].

Some time ago, Cook [4] proposed to measure the ground or excited state (stable or metastable) of two-level a system by means of a short pulse of a probe laser which pumps the transition between the ground state, level 1, and an auxiliary rapidly decaying third level (see Fig. 1). After emission of a photon the atom will be in its ground state, and so it is natural to assume that occurrence of resonance fluorescence means that the atom is in level 1 at the end of the pulse. Absence of resonance fluorescence was assumed to imply that the atom is in level 2. This idea was subsequently used in an experiment of Itano et al. [5] to test the so-called quantum Zeno effect (QZE) [6].

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⁴Lüders stressed its provisional character: "The projection postulate will be employed only until a better understanding of the actual measurement process has been found" (G. Lüders, private communication to G.C.H.).

In a recent paper [7] two of the present authors have investigated to what extent a short probe pulse as in Refs. [4, 5] can be regarded as an effective measurement to which the projection postulate can be applied, at least approximately. We distinguished two cases. If there is no interaction between levels 1 and 2 during a probe pulse, we showed that a single probe pulse does indeed project on either $|1\rangle$ or $|2\rangle$, up to terms decreasing exponentially with its duration. If one has an additional small driving field (Rabi frequency Ω_2) between levels 1 and 2 during a probe pulse the situation is more complicated, and we had to restrict ourselves to *weak* probe pulses (with Rabi frequency Ω_3 much smaller than the Einstein coefficient A_3 of level 3, but still $\Omega_2 \ll \Omega_3$). Under this additional assumption we were able to show that an atom *with* photon emission during the probe pulse is projected onto a state (density matrix) extremely close, but not identical to, $|1\rangle\langle 1|$, and that an atom *without* photon emission is projected onto a state very close to $|2\rangle\langle 2|$. The small difference to the ideal projection-postulate result was explicitly calculated to first order in the small parameters

$$\epsilon_p \equiv \frac{\Omega_2 A_3}{\Omega_3^2}, \quad \epsilon_R \equiv \frac{\Omega_2}{\Omega_3} \quad \text{and} \quad \epsilon_A \equiv \frac{\Omega_2}{A_3}. \quad (1)$$

We also pointed out that these ϵ parameters had to be much smaller than 1 for a pulse of the probe laser to act as an effective measurement. For many probe pulses (“measurements”) this small difference adds up. The cumulative effect was explicitly determined, and we used our results to analyze the experiment of Ref. [5] on the QZE.

The QZE [6] deals with rapidly repeated measurements on a system. Under the assumptions of n ideal measurements, at times $\Delta t, \dots, n\Delta t = t$, t fixed, the QZE predicts a slow-down of the time development and ultimately a freezing of the state for $\Delta t \rightarrow 0$, or $n \rightarrow \infty$. The QZE has found a tremendous interest in the literature [8].

In the experiment of Ref. [5] several thousand ions are stored in a trap and an rf field in resonance is used to drive the transition between two levels 1 and 2, which are essentially stable. The rf field is a so-called π pulse of duration T_π which changes state $|1\rangle$ into state $|2\rangle$. The experiment then endeavors to perform n measurements of the populations of levels 1 and 2 between 0 and T_π by means of n short pulses of a probe laser (see Fig. 2) which pumps the transition between level 1 and the auxiliary rapidly decaying level 3, with occurrence of resonant fluorescence from an atom taken to mean detection of the atom in level 1 and absence thereof as detection of the atom in level 2, as proposed in Ref. [4]. At time T_π the actual population of levels 1 and 2 are then determined through the resonant fluorescence of all atoms in the trap under the influence of a long probe pulse, with the π pulse switched off. The populations determined in this way are in good agreement with the predictions of the QZE.

The relevance of this experiment for the QZE has been contested in the literature. It was pointed out by some authors [9, 10, 11, 12, 13, 14, 15] that there is no need to use the QZE to explain the experiment. One could simply incorporate the auxiliary level and the driving by the probe laser together with the rf field in the Hamiltonian or in the Bloch equations of the corresponding three-level system. From a numerical solution of the Bloch equations one does indeed obtain the final population at time T_π in good agreement with the experiment [11]. The probe pulses are, in this view, not measurements but part of the dynamics.

In our previous paper [7] we used our results on the close, but not perfect, resemblance of individual probe pulses with ideal measurements to determine in closed form, to first order in the small ϵ parameters, the population of level 2 after n probe pulses, at the end of the π pulse. This was compared with a *numerical* solution of the three-level Bloch equations, and an amazing agreement was found. In Ref. [7] our conclusion was that the projection postulate is a useful tool to give a quick and approximate understanding of the type of experiment performed in Ref. [5], but that for a more precise description one needs a more detailed analysis of the actual measurements, as for example carried out by us in Ref. [7].

In the present paper we treat the general case of arbitrary, not necessarily weak, probe pulses. The duration of a probe pulse, τ_p , is assumed to be less than $T_\pi = \pi/\Omega_2$, and our results will be given to first order in the above parameters ϵ_p , ϵ_R and ϵ_A . It will turn out that with increasing Ω_3 , i.e. increasing strength, the probe pulse models an ideal measurement with projection postulate much better than for weak pulses; this is particularly true in the presence of the rf field.

The plan of the paper is as follows. In Section 2 we deal with a single short probe pulse during which the rf field is also switched on, and the description of individual atoms is studied according to whether they do or do not emit photons during a probe pulse. We show that at the end of a probe pulse an atom can only be in one of two states, denoted⁵ by $\tilde{\rho}^>$ and $\tilde{\rho}^0$, respectively, depending on whether or not the atom has emitted photons. These states still contain a contribution from level 3, which will decay during a short transient time. During this decay time the rf field will also change the state, and the state will depend on the particular decay time chosen. We show, however, that there are two uniquely defined states which effectively describe the resultant action at the end of a probe pulse and which one can regard as the projected states of an atom. The two states are independent of the initial state, have no contribution from level 3 and are independent of any particular choice of transient decay time. These uniquely defined states are, however, never exactly realized by the atom and are in this sense virtual and a mathematical tool.

In Section 3 we consider repeated probe pulses in connection with the QZE and the cumulative effect of the corrections to the projection postulate. Closed expressions are given and compared to numerical solutions of the three-level Bloch equations. Our conclusion is that if one would perform the experiment of Ref. [5] with a much stronger probe pulse (with Rabi frequency Ω_3 of the order of the Einstein coefficient A_3) then the result for the level population would be much closer to the case of ideal projection-postulate measurements than for weak probe pulses, as in the original experiment of Ref. [5].

In Section 4 we discuss our results, and in the Appendix we briefly review the quantum jump approach [16, 17, 18, 19] which we use in this paper. The quantum jump approach is essentially equivalent to quantum trajectories [20] and to the Monte-Carlo wave function approach [21].

2. Effect of a single probe pulse on the atomic state

In this Section we discuss how a single probe pulse acts as an effective measurement of an atomic level and to what extent the usual projection postulate can be applied.

The case where the rf field is switched off while the probe pulse is on was discussed in some detail in Ref. [7]. It was shown that if the length of the probe pulse τ_p satisfies the condition

$$\tau_p \gg \max \left\{ A_3^{-1}, A_3/\Omega_3^2 \right\} , \quad (2)$$

then the probe pulse provides an effective reduction of the initial state to $|1\rangle$ or $|2\rangle$ depending whether or not photons were emitted (allowing, in the former case, for a decay of the third-level contributions). The respective probabilities are the same as predicted by the projection postulate for an ideal measurement of the level populations.

If the rf field is switched on during the probe pulse it causes a small pumping between level 1 and 2. However, we will show that, independent of its state at the beginning of a probe pulse, right at the end of the pulse an atom will be in one of two states, $\tilde{\rho}^0$ and $\tilde{\rho}^>$, depending on whether or not it has emitted photons. To allow for the decay of the third-level contributions a transient time at least of the order of several decay times A_3^{-1} is needed. During this time the rf

⁵A tilde on a density matrix indicates that its trace is 1. We will also have occasion to use non-normalized density matrices. Here, a state can be pure or mixed.

field is also active, and therefore the result has a small dependence on the transient time chosen. At the end of this section we therefore mathematically construct two density matrices, $\tilde{\rho}_p^0$ and $\tilde{\rho}_p^>$, in the 1-2 subspace which are independent of the transient time. These states, although not physically realized, can be used in a consistent way as hypothetical or virtual states of an atom at the end of the probe pulse, i.e. as states on which the probe pulse projects.

2.1 Subensemble without photon emission

We now determine the state at the end of the probe pulse for an atom which has not emitted photons. The result is the same as that found in our Ref. [7], but the present derivation is more streamlined. According to Eq. (59) of the Appendix, an atom without photon emission evolves with [16, 18, 19]

$$U_{\text{red}}^I(t, 0) = \exp \left\{ -iH_{\text{red}}^I t / \hbar \right\} \quad (3)$$

where the reduced (or conditional) Hamiltonian is given by Eq. (58) of the Appendix,

$$iH_{\text{red}}^I / \hbar \equiv M = M_0 + M_1 \quad (4)$$

with

$$M_0 = \frac{i}{2} \Omega_3 \{ |3\rangle\langle 1| + |1\rangle\langle 3| \} + \frac{1}{2} A_3 |3\rangle\langle 3| \quad (5)$$

$$M_1 = \frac{i}{2} \Omega_2 \{ |2\rangle\langle 1| + |1\rangle\langle 2| \} . \quad (6)$$

The eigenvalues of M_0 are $\lambda_2^0 = 0$ and

$$\lambda_{1,3}^0 = \frac{1}{4} A_3 \left\{ 1 \pm \sqrt{1 - 4\Omega_3^2/A_3^2} \right\} . \quad (7)$$

For $M = M_0 + M_1$ these go over into λ_i , $i = 1, 2, 3$, which are the roots of the cubic equation

$$p(\lambda) \equiv \det(M - \lambda) = \lambda^3 - \frac{1}{2} A_3 \lambda^2 + \frac{1}{4} (\Omega_2^2 + \Omega_3^2) \lambda - \frac{1}{8} A_3 \Omega_2^2 = 0 . \quad (8)$$

Newton's method or a linear expansion gives⁶

$$\lambda_2 = \frac{1}{2} \Omega_2 \epsilon_p (1 + \mathcal{O}(\epsilon^2)) \quad (9)$$

where $\mathcal{O}(\epsilon^2)$ denotes a correction at least quadratic in $\epsilon \equiv (\epsilon_p, \epsilon_R)$, so that $\mathcal{O}(\epsilon^2)/|\epsilon|^2$ remains bounded. For $\lambda_{1,3}$, one obtains

$$\lambda_{1,3} = \lambda_{1,3}^0 (1 + \mathcal{O}(\epsilon)) . \quad (10)$$

If the λ_i are different (which here means $\lambda_1 \neq \lambda_3$) then M has three normalized eigenvectors $|\lambda_i\rangle$, which are in general not orthogonal. The reciprocal basis vectors $|\lambda^i\rangle$ are defined by

$$\langle \lambda_i | \lambda^j \rangle = \delta_{ij} \quad (11)$$

and satisfy the eigenvalue equation

$$M^\dagger |\lambda^i\rangle = \bar{\lambda}_i |\lambda^i\rangle . \quad (12)$$

⁶In our case λ_2 is real and one has the explicit bounds

$$\frac{1}{2} \Omega_2 \epsilon_p (1 + \epsilon_R^2)^{-1} \leq \lambda_2 \leq \frac{1}{2} \Omega_2 \epsilon_p (1 + \epsilon_p^2) (1 + \epsilon_R^2)^{-1}$$

since $p(\lambda)$ changes sign between these bounds, as seen by insertion.

In particular, an elementary calculation gives

$$|\lambda_2\rangle = -i\epsilon_p|1\rangle + |2\rangle - \epsilon_R|3\rangle + \mathcal{O}(\epsilon^2) \quad (13)$$

$$|\lambda^2\rangle = i\epsilon_p|1\rangle + |2\rangle - \epsilon_R|3\rangle + \mathcal{O}(\epsilon^2) \quad (14)$$

For e^{-Mt} one has the representation⁷

$$e^{-Mt} = \sum_{i=1}^3 e^{-\lambda_i t} |\lambda_i\rangle \langle \lambda^i| = \sum_{i=1}^3 e^{-\lambda_i t} P_i \quad (15)$$

with $P_i \equiv |\lambda_i\rangle \langle \lambda^i|$. If τ_p satisfies the condition of Eq. (2) then the exponentials $\exp\{-\lambda_{1,3}\tau_p\}$ can be neglected, while $\exp\{-\lambda_2\tau_p\} \approx 1$. Therefore for any initial pure state $|\psi\rangle$ one has

$$e^{-M\tau_p}|\psi\rangle = e^{-\lambda_2\tau_p}\langle\lambda^2|\psi\rangle|\lambda_2\rangle + \text{exponentially small terms} \quad (16)$$

which is proportional to $|\lambda_2\rangle$. Thus at the end of a probe pulse the subensemble without photon emissions is, after normalization and up to exponentially small terms, in the state $|\lambda_2\rangle$ and thus independent of the initial state.

The probability $P_0(\tau_p; |\psi\rangle)$ for no emission during a probe pulse, for initial state $|\psi\rangle$ is the norm-squared of Eq. (16). If one has a density matrix $\rho = \sum_i \alpha_i |\psi_i\rangle \langle \psi_i|$ at the beginning of a probe pulse the no-emissions probability becomes

$$P_0(\tau_p; \rho) = \sum_i \alpha_i P_0(\tau_p; |\psi_i\rangle) . \quad (17)$$

By Eqs. (13) and (14) one obtains

$$P_0(\tau_p; \rho) = \rho_{22} - \epsilon_p \pi \frac{\tau_p}{T} \rho_{22} + 2\epsilon_p \text{Im } \rho_{12} - 2\epsilon_R \text{Re } \rho_{23} + \mathcal{O}(\epsilon^2) . \quad (18)$$

We thus arrive at

$$\rho^0(\tau_p; \rho) \equiv e^{-M\tau_p} \rho e^{-M^\dagger\tau_p} = P_0(\tau_p; \rho) |\lambda_2\rangle \langle \lambda_2|$$

and, for the basis $|1\rangle, |2\rangle, |3\rangle$, we obtain from Eq. (13)

$$\tilde{\rho}^0 \equiv |\lambda_2\rangle \langle \lambda_2| = \begin{pmatrix} 0 & -i\epsilon_p & 0 \\ i\epsilon_p & 1 & -\epsilon_R \\ 0 & -\epsilon_R & 0 \end{pmatrix} + \mathcal{O}(\epsilon^2) . \quad (19)$$

We have thus shown that, up to exponentially small terms, a probe pulse projects each atom with no photon emission onto the state $|\lambda_2\rangle$, which is close to $|2\rangle$. The probability for this is given by $P_0(\tau_p; \rho)$ of Eq. (18) where ρ is the density matrix of the atom at the beginning of the probe pulse. In case the *rf* field is switched off ($\Omega_2 = 0$) during the probe pulse, one has $|\lambda_2\rangle = |2\rangle$ and the probe pulse projects onto $|2\rangle$ with probability ρ_{22} , up to exponentially small terms.

2.2 Subensemble with photon emissions

If an atom does emit photons during a probe pulse, the emissions can occur at random times. Therefore, even if one had started with a pure state $|\psi\rangle$ the subensemble of all atoms

⁷Alternatively,

$$e^{-Mt} = \frac{(M - \lambda_2)(M - \lambda_3)}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} e^{-\lambda_1 t} + \text{cyclic permutations} ,$$

which is directly checked by application to the eigenvectors. Comparison with Eq. (15) gives explicit expressions for $|\lambda_i\rangle \langle \lambda^i|$. In the above formula one can take the limit $\lambda_3 \rightarrow \lambda_1$. This leads to a term with the factor $t \exp\{-\lambda_1 t\}$. The following arguments are then easily adapted to this degenerate case.

with emissions is described by a mixture. The corresponding density matrix is denoted by $\rho^>(\tau, \rho)$, with ρ the density matrix at the beginning of a probe pulse, and the normalization of $\rho^>$ is chosen such that $\text{tr } \rho^>(\tau, \rho)$ is the probability for emissions until time τ . The aim of this subsection is to show that $\rho^>(\tau_p, \rho)$ is in general proportional to a ρ -independent matrix. Hence the subensemble with photon emissions is, at the end of a probe pulse, described by a fixed normalized density matrix $\tilde{\rho}^> \equiv \rho^> / \text{tr } \rho^>$, which is independent of the initial state. This density matrix $\tilde{\rho}^>$, which still contains contributions of level 3, will now be explicitly determined.

We denote by $I(\tau'; \rho)$ the probability density for the emission of a photon at time τ' . Now, after a particular atom has emitted its last photon before τ , at time τ' say, the atom is in its ground state $|1\rangle$ and until τ the time development is given by

$$e^{-M(\tau-\tau')}|1\rangle\langle 1|e^{-M^\dagger(\tau-\tau')} \equiv \rho^0(\tau - \tau'; |1\rangle) \quad (20)$$

Hence the subensemble is described by

$$\begin{aligned} \rho^>(\tau; \rho) &= \int_0^\tau d\tau' I(\tau'; \rho) \rho^0(\tau - \tau'; |1\rangle), \\ &= \int_0^\tau d\tau' I(\tau - \tau'; \rho) \rho^0(\tau'; |1\rangle). \end{aligned} \quad (21)$$

This is also seen directly from Eq. (66) of the Appendix. The density matrix of the complete ensemble is given by

$$\rho(\tau; \rho) = \rho^0(\tau; \rho) + \rho^>(\tau; \rho), \quad (22)$$

and

$$\text{tr } \rho^>(\tau; \rho) = 1 - \text{tr } \rho^0(\tau; \rho) = 1 - P_0(\tau; \rho). \quad (23)$$

Since in our case $I(\tau; \rho) = A_3 \rho_{33}(\tau; \rho)$ by Eq. (69) of the Appendix, Eqs. (21) and (22) give the integral equation

$$I(\tau; \rho) = A_3 \rho_{33}^0(\tau; \rho) + A_3 \int_0^\tau d\tau' I(\tau - \tau'; \rho) \rho_{33}^0(\tau'; |1\rangle). \quad (24)$$

This can be solved by Laplace transform⁸. Since, by Eq. (20), ρ_{33}^0 is a sum of exponential terms whose exponents do not depend on ρ , I is of the form

$$I(\tau; \rho) = c_0 + c_1 e^{-\mu_1 \tau} + \sum_{\alpha=2}^9 c_\alpha e^{-\mu_\alpha \tau} \quad (25)$$

where $c_i = c_i(\rho)$. One has $\text{Re } \mu_i > 0$, μ_1 is real and of the order of λ_2 , while all the other μ_α 's are of the order of λ_1 and λ_3 ⁹.

Therefore one sees that c_0 is the stationary emission rate of the three-level system. Furthermore, considering times of the order of the length of the probe pulse, i.e. $\exp\{-\mu_1 \tau\} \approx 1$ and $\text{Re } \mu_\alpha \tau \gg 1$ for $\alpha \geq 2$, one sees that $c_0 + c_1$ must be positive. Physically the emission rate for such times cannot exceed the stationary emission rate $A_3 \Omega_3^2 / (A_3^2 + 2\Omega_3^2)$ of the 1-3 system with $\Omega_2 = 0$ and initial state $|1\rangle$, and the same is true for very large τ , for which $\exp\{-\mu_1 \tau\} \ll 1$. Hence one has the inequalities

$$0 \leq \left\{ \begin{array}{c} c_0 \\ c_0 + c_1(\rho) \end{array} \right\} \leq \frac{A_3 \Omega_3^2}{A_3^2 + 2\Omega_3^2}. \quad (26)$$

⁸Alternatively one could solve the Bloch equations for ρ .

⁹For $\Omega_2 = 0$ one has $\mu_1 = 0$, and in general μ_1 is quadratic in Ω_2 . The μ_i 's are in fact the eigenvalues of the matrix in the Bloch equations. Two of the eigenvalues vanish for $\Omega_2 = 0$. From Eq. (24) one can determine μ_1 explicitly as $\mu_1 = 2\epsilon_p \Omega_2 (A_3^2 + \Omega_3^2) / (A_3^2 + 2\Omega_3^2) + \mathcal{O}(\Omega_2^4)$.

Therefrom one obtains

$$c_0, |c_1(\rho)| \leq \frac{A_3 \Omega_3^2}{A_3^2 + 2\Omega_3^2}. \quad (27)$$

Only these rough inequalities will be needed¹⁰.

From Eq. (15) for e^{-Mt} one obtains

$$\rho^0(\tau; |1\rangle) = \sum_{i,j} P_i |1\rangle \langle 1| P_j^\dagger e^{-(\lambda_i + \bar{\lambda}_j)\tau} \quad (28)$$

where P_i was defined as $|\lambda_i\rangle \langle \lambda_i|$. Inserting Eqs. (25) and (28) into Eq. (21) with $\tau = \tau_p$ gives

$$\begin{aligned} \rho^>(\tau_p; \rho) &= (c_0 + c_1 e^{-\mu_1 \tau_p}) \int_0^{\tau_p} d\tau' \rho^0(\tau'; |1\rangle) \\ &+ c_1 \int_0^{\tau_p} d\tau' \left(e^{-\mu_1(\tau_p - \tau')} - e^{-\mu_1 \tau_p} \right) \rho^0(\tau'; |1\rangle) \\ &+ \int_0^{\tau_p} d\tau' \sum_{\alpha \geq 2} c_\alpha e^{-\mu_\alpha(\tau_p - \tau')} e^{-2\lambda_2 \tau'} P_2 |1\rangle \langle 1| P_2^\dagger \\ &+ \int_0^{\tau_p} d\tau' \sum_{\alpha \geq 2} c_\alpha e^{-\mu_\alpha(\tau_p - \tau')} \sum_{(i,j) \neq (2,2)} e^{-(\lambda_i + \bar{\lambda}_j)\tau'} P_i |1\rangle \langle 1| P_j^\dagger. \end{aligned} \quad (29)$$

It will be shown that the third and the last term are of order ϵ^2 . The last term is clearly proportional to exponentially small terms and can therefore be omitted. Since $e^{-\mu_1(\tau_p - \tau')} - e^{-\mu_1 \tau_p} \leq \mu_1 \tau'$, the second term is bounded by

$$\begin{aligned} c_1 \mu_1 \int_0^{\tau_p} d\tau' \tau' \rho^0(\tau'; |1\rangle) &= c_1 \mu_1 \int_0^{\tau_p} d\tau' \tau' e^{-2\lambda_2 \tau'} P_2 |1\rangle \langle 1| P_2^\dagger \\ &+ c_1 \mu_1 \sum_{(i,j) \neq (2,2)} \int_0^{\tau_p} d\tau' \tau' e^{-(\lambda_i + \bar{\lambda}_j)\tau'} P_i |1\rangle \langle 1| P_j^\dagger. \end{aligned} \quad (30)$$

Now, $P_2 |1\rangle = |\lambda_2\rangle \langle \lambda_2| 1\rangle$ is of order ϵ , by Eq. (14), and $P_2 |1\rangle \langle 1| P_2^\dagger$ is thus of order¹¹ ϵ^2 . The second integral in Eq. (30) is proportional to $(\lambda_i + \bar{\lambda}_j)^{-2}$. Multiplied by $c_1 \mu_1$ this becomes bounded by ϵ^2 . Thus the second term in Eq. (29) is bounded by ϵ^2 . For the third term the argument is again based on $P_2 |1\rangle \langle 1| P_2^\dagger = \mathcal{O}(\epsilon^2)$. Hence

$$\rho^>(\tau_p; \rho) = (c_0 + c_1 e^{-\mu_1 \tau_p}) \int_0^{\tau_p} d\tau' \rho^0(\tau'; |1\rangle) + \mathcal{O}(\epsilon^2) \quad (31)$$

and thus one has for the normalized density matrix

$$\tilde{\rho}^>(\tau_p; \rho) = \tilde{\rho}^>(\tau_p) \propto \int_0^{\tau_p} d\tau' \rho^0(\tau'; |1\rangle) + \mathcal{O}(\epsilon^2) \quad (32)$$

which is, up to order ϵ^2 , independent of the initial state ρ (cf. footnote 11).

¹⁰From Eq. (24) one obtains by a more detailed calculation

$$\begin{aligned} c_0 &= \frac{1}{2} \frac{A_3 \Omega_3^2}{A_3^2 + \Omega_3^2} \\ c_1(\rho) &= c_0 \left(\frac{A_3^2}{A_3^2 + 2\Omega_3^2} - 2 \frac{A_3^2 + \Omega_3^2}{A_3^2 + 2\Omega_3^2} \langle \lambda_2 | \rho | \lambda_2 \rangle \right). \end{aligned}$$

¹¹ Normalization of $\rho^>$ involves division by $\text{tr} \rho^> = 1 - P_0(\tau_p, \rho)$. For ρ close to $|\lambda_2\rangle \langle \lambda_2|$ this is of order ϵ , and in this case Eq. (32) holds only up to order $\epsilon_p T_p / \tau_p$. In the subsequent applications, however, this will play no role.

The integral can be computed in closed form by choosing for ρ in Eqs. (21) and (22) the stationary state ρ^{ss} of the three-level system driven by Ω_3 and Ω_2 . Then the emission rate $I = A_3 \rho_{33}^{ss}$ is time independent, and one obtains

$$\rho^{ss} = e^{-M\tau_p} \rho^{ss} e^{-M^\dagger \tau_p} + A_3 \rho_{33}^{ss} \int_0^{\tau_p} d\tau' \rho^0(\tau'; |1\rangle) . \quad (33)$$

With Eq. (32) this yields

$$\tilde{\rho}^>(\tau_p) = \left(\rho^{ss} - e^{-M\tau_p} \rho^{ss} e^{-M^\dagger \tau_p} \right) / \text{tr}(\cdot) + \mathcal{O}(\epsilon^2) . \quad (34)$$

One can express ρ^{ss} directly in terms of M as¹²

$$\rho^{ss} = (M - \frac{1}{2}A_3)(M^\dagger - \frac{1}{2}A_3) / \text{tr}(\cdot) . \quad (35)$$

One can understand Eq. (34) as follows. It takes much longer than the time τ_p to reach the stationary three-level state, and therefore there is not enough time to build up an appreciable population of level 2. With the second term one just subtracts the excess population of this level from ρ^{ss} because

$$e^{-M\tau_p} \rho^{ss} e^{-M^\dagger \tau_p} = e^{-2\lambda_2 \tau_p} \langle \lambda^2 | \rho^{ss} | \lambda^2 \rangle | \lambda_2 \rangle \langle \lambda_2 | + \text{exponentially small terms} . \quad (36)$$

From Eqs. (34)-(36) one can now calculate in a straightforward way the normalized density matrix $\tilde{\rho}^>(\tau_p)$ at the end of a probe pulse for the subensemble *with* photon emissions and obtains

$$\tilde{\rho}^> = \frac{1}{A_3^2 + 2\Omega_3^2 + A_3^2 \epsilon_p \Omega_2 \tau_p} \begin{pmatrix} A_3^2 + \Omega_3^2 & i\epsilon_p A_3^2 & iA_3 \Omega_3 \\ -i\epsilon_p A_3^2 & A_3^2 \epsilon_p \Omega_2 \tau_p & \epsilon_R (A_3^2 + \Omega_3^2) \\ -iA_3 \Omega_3 & \epsilon_R (A_3^2 + \Omega_3^2) & \Omega_3^2 \end{pmatrix} + \mathcal{O}(\epsilon^2) . \quad (37)$$

It is noteworthy that this is just, except for the ϵ terms, the two-level stationary state of the 1-3 system (with $\Omega_2 = 0$). In addition $\tilde{\rho}^>$ has a 22-component proportional to τ_p . This results from the possibility of macroscopic dark periods for the V system under consideration [22] so that the emission of photons may stop before the probe pulse ends. Such atoms are then in a state close to $|2\rangle$ [16] and contribute to $\tilde{\rho}_{22}^>$.

The effect of the probe pulse is therefore that right at its end an atom, which is initially described by ρ , is projected with probability $1 - P_0(\tau_p; \rho)$ onto $\tilde{\rho}^>(\tau_p)$ and with probability $P_0(\tau_p; \rho)$ onto $\tilde{\rho}^0$. We note that $\tilde{\rho}^>$ and $\tilde{\rho}^0$ still have a 33 component which is small for $\tilde{\rho}^0$ but can be appreciable for $\tilde{\rho}^>$ and whose decay will now be studied.

2.3 Decay of the 33 component and the question of measurement by a probe pulse

At the end of a probe pulse one still has $\Omega_2 \neq 0$ while $\Omega_3 = 0$. The time development of an arbitrary density matrix is then governed by the corresponding Bloch equations with $\Omega_3 = 0$,

¹²From Eqs. (5) and (6) one finds $(M - \frac{1}{2}A_3)|3\rangle = \frac{i}{2}\Omega_3|1\rangle$ and $M + M^\dagger = A_3|3\rangle\langle 3|$. From this one readily obtains

$$\begin{aligned} & \int_0^{\tau_p} d\tau' e^{-M\tau'} |1\rangle\langle 1| e^{-M^\dagger \tau'} \\ &= -\frac{4}{\Omega_3^2 A_3} \left(M - \frac{1}{2}A_3 \right) \int_0^{\tau_p} d\tau' \frac{d}{d\tau'} \left(e^{-M\tau'} e^{-M^\dagger \tau'} \right) \left(M^\dagger - \frac{1}{2}A_3 \right) \\ &= \frac{4}{\Omega_3^2 A_3} \left(M - \frac{1}{2}A_3 \right) \left(1 - e^{-M\tau_p} e^{-M^\dagger \tau_p} \right) \left(M^\dagger - \frac{1}{2}A_3 \right) . \end{aligned}$$

Comparison with Eqs. (33) and (34) gives (35).

leading to a rapid decay of all $i3$ and $3i$ components, $i = 1, 2, 3$. Simultaneously the 1-2 transition is weakly pumped by Ω_2 . If $\tilde{\rho}_{33}^>$ is very small the net effect of this is the same as increasing $\tilde{\rho}_{11}^>$ by $\tilde{\rho}_{33}^>$ and replacing all $\tilde{\rho}_{i3}^>$ and $\tilde{\rho}_{3i}^>$ by 0 right after the probe pulse. This was the case in Ref. [7]. If, however, $\tilde{\rho}_{33}^>$ is not small, then the pumping from 1 to 2 during the decay is not quite the same with this replacement.

Without employing Bloch equations one can very easily determine the time development by means of Eqs. (64) and (66). Between probe pulses the *reduced* time development operator is given by $\exp\{-M_b\tau\}$ with

$$M_b = \frac{i}{2} \Omega_2 \{|2\rangle\langle 1| + |1\rangle\langle 2|\} + \frac{1}{2} A_3 |3\rangle\langle 3|.$$

Hence, with

$$U_\pi(\tau + \tau_0, \tau_0) \equiv \begin{pmatrix} \cos \frac{1}{2} \Omega_2 \tau & -i \sin \frac{1}{2} \Omega_2 \tau & 0 \\ -i \sin \frac{1}{2} \Omega_2 \tau & \cos \frac{1}{2} \Omega_2 \tau & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv U_\pi(\tau) \quad (38)$$

and with the projector $P_{12} \equiv |1\rangle\langle 1| + |2\rangle\langle 2|$ one has

$$e^{-M_b\tau} = U_\pi(\tau) P_{12} + e^{-\frac{1}{2} A_3 \tau} |3\rangle\langle 3| \quad (39)$$

where U_π describes the 1-2 pumping by the rf field and the last term the rapid decay of level 3. Until time $\tau_p + \tau$ the density matrix $\tilde{\rho}^>(\tau_p)$ has then developed, by Eqs. (63), (59) and (66) of the Appendix, or by the analog of Eq. (21), to

$$\begin{aligned} \tilde{\rho}^>(\tau_p + \tau) &= e^{-M_b\tau} \tilde{\rho}^>(\tau_p) e^{-M_b^\dagger \tau} \\ &+ \int_0^\tau d\tau' I(\tau'; \tilde{\rho}^>(\tau_p)) e^{-M_b(\tau-\tau')} |1\rangle\langle 1| e^{-M_b^\dagger(\tau-\tau')} \end{aligned} \quad (40)$$

where, from Eq. (39),

$$I(\tau'; \tilde{\rho}^>(\tau_p)) = A_3 e^{-A_3 \tau'} \tilde{\rho}_{33}^>(\tau_p).$$

An analogous equation holds for the subensemble $\tilde{\rho}^0(\tau_p + \tau)$ of Subsection 2.1. During the decay of the 33 component an additional photon may be emitted. Eq. (40) is easily evaluated for times τ larger than a transient time τ_{tr} for which $\exp\{-\frac{1}{2} A_3 \tau_{tr}\}$ is at most of the order of ϵ^2 and thus can be neglected. E.g. for

$$\tau_{tr} \cong 15/A_3 \quad (41)$$

the exponential is less than 10^{-4} . Eq. (40) becomes, for $\tau \geq \tau_{tr}$,

$$\begin{aligned} \tilde{\rho}^>(\tau_p + \tau) &= U_\pi(\tau) \left\{ P_{12} \tilde{\rho}^>(\tau_p) P_{12} \right. \\ &\left. + \tilde{\rho}_{33}^>(\tau_p) \int_0^\tau d\tau' A_3 e^{-A_3 \tau'} U_\pi^\dagger(\tau') |1\rangle\langle 1| U_\pi(\tau') \right\} U_\pi^\dagger(\tau). \end{aligned} \quad (42)$$

With Eq. (38) for U_π the integral is readily evaluated. Denoting the matrix in the brackets by $\tilde{\rho}_P^>$, one obtains

$$\tilde{\rho}_P^> = \begin{pmatrix} \tilde{\rho}_{11}^>(\tau_p) + \tilde{\rho}_{33}^>(\tau_p) & \tilde{\rho}_{12}^>(\tau_p) - \frac{i}{2} \tilde{\rho}_{33}^>(\tau_p) \epsilon_A & 0 \\ \tilde{\rho}_{21}^>(\tau_p) + \frac{i}{2} \tilde{\rho}_{33}^>(\tau_p) \epsilon_A & \tilde{\rho}_{22}^>(\tau_p) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (43)$$

Insertion of $\tilde{\rho}^>(\tau_p)$ leads to

$$\tilde{\rho}_P^> = \frac{1}{A_3^2 + 2\Omega_3^2 + A_3^2 \epsilon_p \Omega_2 \tau_p} \begin{pmatrix} A_3^2 + 2\Omega_3^2 & i\epsilon_p A_3^2 - \frac{i}{2} \epsilon_A \Omega_3^2 & 0 \\ -i\epsilon_p A_3^2 + \frac{i}{2} \epsilon_A \Omega_3^2 & A_3^2 \epsilon_p \Omega_2 \tau_p & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (44)$$

and thus, for $\tau \geq \tau_{\text{tr}}$,

$$\tilde{\rho}^>(\tau_p + \tau) = U_\pi(\tau) \tilde{\rho}_p^> U_\pi^\dagger(\tau) \quad (45)$$

up to the same order in ϵ as $\tilde{\rho}^>(\tau_p)$ (i.e. up to order ϵ^2 , unless one had started close to $|2\rangle$ at the beginning of the probe pulse).

In a similar way, one finds for the normalized density matrix of the subensemble without emissions during the probe pulse the density matrix at time $\tau_p + \tau$, for $\tau > \tau_{\text{tr}}$,

$$\tilde{\rho}^0(\tau_p + \tau) = U_\pi(\tau) \tilde{\rho}_p^0 U_\pi^\dagger(\tau) \quad (46)$$

with

$$\tilde{\rho}_p^0 = \begin{pmatrix} 0 & -i\epsilon_p & 0 \\ i\epsilon_p & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (47)$$

again up to the same order in ϵ as $\tilde{\rho}^0(\tau_p)$.

These results can be viewed in the following way. At the end of a probe pulse both the subensemble with and without photon emissions still have components involving the third level, which subsequently decay during the transient time. Simultaneously the population of level 1 is increased correspondingly. In addition, during this decay, one also has the action of the rf field which introduces a change dependent on the length of the transient decay time one considers. However, from Eqs. (45) and (46) it is apparent that for times larger than the transient time one obtains the correct result also if one projects onto the states $\tilde{\rho}_p^>$ and $\tilde{\rho}_p^0$ at the end of a probe pulse and then develops with the rf field only. These projection states can be considered virtual in the sense that they are really never quite realized in the actual time development of the atom. But it is at least *formally* consistent to say that a probe pulse acts as an effective measurement and projects onto $\tilde{\rho}_p^>$ or $\tilde{\rho}_p^0$.

If one neglects all ϵ terms the states $\tilde{\rho}_p^>$ and $\tilde{\rho}_p^0$ become $|1\rangle\langle 1|$ and $|2\rangle\langle 2|$, respectively, as for an ideal measurement to which the projection postulate can be applied. The ϵ terms can thus be viewed as corrections to the projection postulate. In the case $\epsilon_A \ll 1$ ($\Omega_2 \ll A_3$) the second term in the 1-2 component of $\tilde{\rho}_p^>$ can be neglected, and $\tilde{\rho}_p^>$ then reduces to Eq. (61) of Ref. [7].

3. Applications to the Quantum Zeno effect: Corrections to the projection-postulate results

As shown above a probe pulse, regarded as a measurement pulse, models very closely an ideal measurement to which the projection postulate can be applied. As pointed out in Ref. [7], for repeated probe pulses the small differences may add up, however, and the net result for an ensemble of atoms was calculated there in the case $\Omega_3 \ll A_3$. Here this will now be calculated for an ensemble of atoms without this restriction for n probe pulses during a π pulse of length $T_\pi = \pi/\Omega_2$ (cf. Fig. 2). The time between two pulses is denoted by ΔT . It will be assumed that $\Delta T \gg A_3^{-1}$, in fact ΔT larger than the transient decay time τ_{tr} will do.

At the end of the k -th probe pulse the total ensemble of atoms consists of two subensembles, one with and the other without emissions during the k -th pulse. Hence, by Eqs. (37) and (19), the total density matrix $\rho(t)$ at time $t = k \cdot (\Delta T + \tau_p)$ is of the form

$$\rho(t) = \alpha(k) \tilde{\rho}^> + \beta(k) \tilde{\rho}^0 + \mathcal{O}(\epsilon^2) \quad (48)$$

with $\alpha + \beta = 1$. We will now determine $\alpha(k)$ and $\beta(k)$.

We consider an ensemble with density matrix $\tilde{\rho}^>$ at the end of a probe pulse, develop for the time ΔT without the probe pulse and then again switch on the next probe pulse. We define p as the probability of finding no photons during this pulse. Similarly, q is defined as the no-photon probability for starting with $\tilde{\rho}^0$ instead of $\tilde{\rho}^>$. We will calculate p and q explicitly further below.

Now, if the density matrix is given by Eq. (48) at the end of the k -th probe pulse, then at the end of the next pulse it is given by

$$(1-p)\alpha(k)\tilde{\rho}^> + p\alpha(k)\tilde{\rho}^0 + (1-q)\beta(k)\tilde{\rho}^> + q\beta(k)\tilde{\rho}^0. \quad (49)$$

Hence β satisfies the recursion relation

$$\beta(k+1) = p\alpha(k) + q\beta(k). \quad (50)$$

Using $\alpha = 1 - \beta$ the solution is seen to be

$$\beta(k) = p \frac{1 - (q-p)^{k-1}}{1 - (q-p)} + (q-p)^{k-1}\beta(1) \quad (51)$$

for $k > 1$. We determine $\beta(1)$ for the case that at $t = 0$ all atoms are prepared in the ground state. At the beginning of the first probe pulse the state is then $U_\pi(\Delta T)|1\rangle$, and $\beta(1)$ is the probability for no photon emission until the end of the first pulse. With Eqs. (38) and (18) and the abbreviations

$$c \equiv c(\Delta T) \equiv \cos(\Omega_2 \Delta T), \quad s \equiv s(\Delta T) \equiv \sin(\Omega_2 \Delta T) \quad (52)$$

one easily obtains

$$\beta(1) = \frac{1}{2}(1-c) + \epsilon_p s - \frac{1}{2} \pi \frac{\tau_p}{T_\pi} (1-c)\epsilon_p + \mathcal{O}(\epsilon^2). \quad (53)$$

In the determination of the no-photon probabilities p and q the relevant density matrices at the beginning of the next probe pulse are $U_\pi(\Delta T) \tilde{\rho}_P^> U_\pi(\Delta T)^\dagger$ and $U_\pi(\Delta T) \tilde{\rho}_P^0 U_\pi(\Delta T)^\dagger$. Using Eq. (18) one obtains, with $\epsilon_A = \Omega_2/A_3$,

$$\begin{aligned} p &= \frac{1}{2}(1-c) + \epsilon_p \left\{ 2s \frac{A_3^2 + \Omega_3^2}{A_3^2 + 2\Omega_3^2} + \frac{1}{2} \Omega_2 \tau_p c \frac{3A_3^2 + 2\Omega_3^2}{A_3^2 + 2\Omega_3^2} - \frac{1}{2} \Omega_2 \tau_p \right\} \\ &\quad - \frac{1}{2} s \frac{\Omega_3^2}{A_3^2 + 2\Omega_3^2} \epsilon_A + \mathcal{O}(\epsilon^2) \\ q &= \frac{1}{2}(1+c) - \epsilon_p \left\{ 2s + \frac{1}{2} \Omega_2 \tau_p (1+c) \right\} + \mathcal{O}(\epsilon^2). \end{aligned} \quad (54)$$

One can now insert $\beta(k)$ of Eq. (51) and $\alpha(k) = 1 - \beta(k)$ with $k = n$ into Eq. (48) to obtain the density matrix $\rho(t = T_\pi)$ at the end of the π pulse, after n probe pulses. Expanding in terms of ϵ one obtains for the population of level 2

$$\begin{aligned} \rho_{22}(T_\pi) &= \frac{1}{2}(1-c^n) + \epsilon_p \left\{ s c^{n-1} \frac{(2n-1)A_3^2 + 3n\Omega_3^2}{A_3^2 + 2\Omega_3^2} \right. \\ &\quad \left. + \pi \frac{\tau_p}{T_\pi} n c^n \frac{A_3^2 + \Omega_3^2}{A_3^2 + 2\Omega_3^2} - \frac{1-c^n}{1-c} \left(s + \pi \frac{\tau_p}{T_\pi} \right) \frac{\Omega_3^2}{A_3^2 + 2\Omega_3^2} \right\} \\ &\quad - \frac{1}{4} \epsilon_A s \frac{\Omega_3^2}{A_3^2 + 2\Omega_3^2} \left\{ \frac{1-c^{n-1}}{1-c} + (n-1)c^{n-1} \right\} + \mathcal{O}(\epsilon^2) \end{aligned} \quad (55)$$

where $c(\Delta T)$ and $s(\Delta T)$ are given by Eq. (52), $\epsilon_p = A_3 \Omega_2 / \Omega_3^2 \ll 1$ and $\epsilon_A = \Omega_2 / A_3 \ll 1$.

This “quantum jump” result contains additional ϵ terms when compared with the special case in Eq. (76) of Ref. [7] for $\Omega_3^2 \ll A_3^2$, while up to zeroth order it is the same. The zeroth order gives the approximation

$$\rho_{22}(T_\pi) \cong \frac{1}{2} \left\{ 1 - \cos^n \left(\frac{1}{n} - \frac{\tau_p}{T_\pi} \right) \right\} \quad (56)$$

Projection Postulate					
n	$\Delta T = T_\pi/n$	$\Delta T = T_\pi/n - \tau_p$	Quantum Jump	Bloch eq.	Observed [5]
1	1.00000	0.99978	0.99978	0.99978	0.995
2	0.50000	0.49957	0.49960	0.49960	0.500
4	0.37500	0.35985	0.36062	0.36056	0.335
8	0.23460	0.20857	0.20998	0.20993	0.194
16	0.13343	0.10029	0.10215	0.10212	0.103
32	0.07156	0.03642	0.03841	0.03840	0.013
64	0.00371	0.00613	0.00789	0.00789	-0.006

Table 1: Predicted and observed population of level 2 at the end of the π pulse for n probe pulses of length τ_p .

Projection Postulate			
n	$\Delta T = T_\pi/n - \tau_p$	Quantum Jump	Bloch eq.
1	0.99978	0.99978	0.99978
2	0.49957	0.49957	0.49956
4	0.35985	0.35985	0.35979
8	0.20857	0.20858	0.20853
16	0.10029	0.10030	0.10027
32	0.03642	0.03642	0.03641
64	0.00613	0.00613	0.00613

Table 2: Predicted population of level 2 at the end of the π pulse for the parameters of the experiment, but with $\Omega_3 = A_3/2$.

while for n *instantaneous* ideal measurements the projection postulate would yield

$$\frac{1}{2} \left\{ 1 - \cos^n \frac{\pi}{n} \right\}. \quad (57)$$

Thus the zeroth order result in Eq. (56) can be viewed as the result of the projection postulate with the finite duration of the measurement pulse taken into account. The intuitive reason for this has been discussed in some detail in Ref. [7].

In the following tables we have evaluated the expressions for $\rho_{22}(T_\pi)$ of Eq. (56), i.e. the (modified) projection-postulate result, and of Eq. (55), the quantum jump result, as well as a numerical solution of the corresponding Bloch equations of Eq. (70) of the Appendix.

Table 1 is calculated for the parameters of Ref. [5], i.e. $T_\pi = 0.256$ s, $A_3 = 1.2 \cdot 10^8$ /s, $\Omega_3 = 1.9 \cdot 10^6$ /s, and $\tau_p = 2.4 \cdot 10^{-3}$ s; here $\Omega_3^2 \ll A_3^2$. The column with the quantum-jump results coincides with the corresponding column in Table 2 of Ref. [7]. This shows that the additional ϵ terms in Eq. (55) compared to Eq. (76) of Ref. [7] are indeed negligible in this case. The agreement between the quantum-jump results and Bloch equations is excellent, while the column with the (modified) projection-postulate results show the small, but still markedly noticeable, relevance of the ϵ terms in the quantum-jump expression of Eq. (55). In this case the probe or measurement pulses give a very good, but not perfect, realization of ideal measurements.

A much better realization of ideal measurements can be obtained by choosing Ω_3 much larger, e.g. of the order of A_3 . Then one has a strong pumping between levels 1 and 3, with many photon emissions, and the last photon during a probe pulse will therefore be emitted shortly

before its end. After this the atom is in the ground state, and the rf field has little time to build up a correlation between levels 1 and 2. On the other hand, the population of level 3 at the end of a probe pulse grows with Ω_3^2 , and during the transient decay time after the end of a probe this leads to a build-up of the 1-2 correlation, but with the opposite sign, as seen from $\tilde{\rho}_P^>$ in Eq. (44). The net result is that the probe pulse projects the subensemble with photon emissions onto a state much closer to $|1\rangle\langle 1|$ than for small Ω_3 ¹³. For large Ω_3 one therefore expects the projection-postulate results to be much closer to the quantum jump or Bloch equations results. This is indeed borne out in Table 2 for $\Omega_3 = A_3/2$, the other parameters being as in Table 1. Now the projection postulate results (with finite pulse duration) in the first column agree with the other ones much better, with the difference starting in the fifth decimal.

4. Discussion

In this paper we have investigated in detail to what extent a short pulse of a probe laser, which pumps the ground state of an atom to a third auxiliary level, can be regarded as a measurement of the states of the two-level system. In contrast to our previous paper [7] we have now allowed an arbitrary strength of the probe pulse. We have shown that in this general case the projection-postulate result is modified by additional small correction terms. But we have also shown that for a strong probe pulse, e.g. Ω_3 in the order of A_3 , the probe pulse acts in a way much closer to the ideal projection postulate. This is evident from the “virtual” projection matrices $\tilde{\rho}_P^>$ and $\tilde{\rho}_P^0$ of Eqs. (45) and (46) which for large Ω_3 are extremely close to $|1\rangle\langle 1|$ and $|2\rangle\langle 2|$, respectively. Indeed, the off-diagonal elements, which are already small for weak probe pulses, become orders of magnitude smaller for strong probe pulses.

The slow-down of the time development of a state under repeated probe pulses (“measurements”) is apparent in Tables 1 and 2. For strong probe pulses this is much closer to the projection-postulate predictions (with finite pulse duration taken into account). But an actual freezing of the state, as predicted by the projection postulate for instantaneous ideal measurement for $\Delta t \rightarrow 0$, is of course still not obtainable by these probe pulses. If one decreases the time ΔT between the pulse to the order of the transient time τ_{tr} or even to A_3^{-1} , then the third level does not decay completely to level 1, and the probe pulse can no longer be regarded as a measurement pulse.

Appendix: The quantum jump approach in quantum optics

We briefly summarize the quantum jump approach used in this paper. The quantum jump approach [16, 17, 18, 19], quantum trajectories [20] and the Monte-Carlo wave function approach [21] are essentially equivalent. It describes a radiating atom between photon detections by a reduced (or conditional) time evolution operator giving the time development under the condition that no photon has been detected [16]. After a photon detection one has to reset the atom to the reset state (“jump”), with ensuing reduced time development, and so on. The general reset states have been determined in Ref. [18]; cf. also Ref. [19]. For a driven system with many emissions one then obtains a stochastic path, also called a quantum trajectory [20]. For a V system as considered in this paper the reset state after an emission is the ground state. The reduced time development together with the reset states provide a complete stochastic description of the time development of the atom [18, 19]. Starting with this description one can then derive the Bloch equations describing an ensemble of radiating atoms [16, 18].

We consider the V system depicted in Fig. 1. In this system the upper levels 2 and 3 couple to a common ground level 1, with Einstein coefficient A_3 (in this paper level 2 is taken as

¹³Note that $\epsilon_P = \Omega_2 A_3 / \Omega_3^2$ decreases quadratically with Ω_3 . Hence the off-diagonal elements in $\rho_P^>$ and ρ_P^0 become extremely small when Ω_3 is increased to the order of A_3 .

stable). We assume here that $\omega_{32} \equiv \omega_3 - \omega_2$ is in the optical range. For simplicity we consider zero detunings of the driving fields, whose (real) Rabi frequencies are denoted by Ω_2 and Ω_3 , respectively. In the interaction picture with respect to the free atomic Hamiltonian H_0^A , the reduced Hamiltonian H_{red}^I is given by [7, 16, 19]

$$H_{\text{red}}^I/\hbar = \frac{1}{2} \begin{pmatrix} 0 & \Omega_2 & \Omega_3 \\ \Omega_2 & 0 & 0 \\ \Omega_3 & 0 & -iA_3 \end{pmatrix} \equiv -iM. \quad (58)$$

where the atomic operator M is defined by the l.h.s. and where we have used matrix notation with respect to the atomic basis $|1\rangle, |2\rangle, |3\rangle$. The time development of an atom between emissions is then given by

$$U_{\text{red}}^I(\tau) = e^{-iH_{\text{red}}^I\tau/\hbar} = e^{-M\tau}. \quad (59)$$

The no-photon probability until time τ is then, for initial state $|\psi\rangle$,

$$P_0(\tau; |\psi\rangle) = \|e^{-M\tau}|\psi\rangle\|^2 \quad (60)$$

or, more generally for an initial density matrix ρ ,

$$P_0(\tau; \rho) = \text{tr} \left\{ e^{-M\tau} \rho e^{-M^\dagger\tau} \right\}. \quad (61)$$

The probability for the first photon to be emitted in $(\tau, \tau + d\tau)$ equals $P_0(\tau; \rho) - P_0(\tau + d\tau; \rho) \equiv w_1(\tau; \rho)d\tau$, where

$$w_1(\tau; \rho) = -\frac{d}{d\tau} P_0(\tau; \rho) \quad (62)$$

is the probability density for the first photon¹⁴. For small upper level separation nonzero off-diagonal generalized damping terms may appear which lead to interesting coherence effects [23, 24, 25, 26]. For general n -level systems the reduced Hamiltonian is derived in Ref. [18, 19].

The reduced time development is not unitary. The reason is that it does not describe the time evolution of the whole ensemble but that of the subensemble with no photons. The size of this subensemble is decreasing in time since an atom for which a photon has been detected leaves the subensemble, and this is reflected in the decrease of the norm squared in Eq. (60). The above probability density determines the (random) time for the first photon. After that the atom is reset, for a V system to the ground state, $|1\rangle$. The next emission time is then determined by $w_1(\tau; |1\rangle)$, and so on. In this way one obtains a quantum trajectory.

From this description of single systems one can recover the usual Bloch equations of the complete ensemble as follows [18]. The density matrix $\rho(\tau)$ of the ensemble is a sum of two terms, $\rho^>$ and ρ^0 , corresponding to a subensemble of atoms with and without photon emissions until time τ , respectively,

$$\rho(\tau) = \rho^0(\tau) + \rho^>(\tau). \quad (63)$$

From Eq. (59) one has for initial state ρ

$$\rho^0(\tau; \rho) = e^{-M\tau} \rho e^{-M^\dagger\tau}. \quad (64)$$

If $I(\tau'; \rho)d\tau'$ denotes the (unconditioned) probability to find a photon in $(\tau', \tau' + d\tau')$ then, at time τ , the sub-subensemble of atoms with their last emission in this interval is described by

$$I(\tau'; \rho)d\tau' \rho^0(\tau - \tau'; |1\rangle) \quad (65)$$

¹⁴Depending on the parameters there may be a finite probability that no photon is emitted at all. Therefore this probability density need not be normalized to 1.

and therefore

$$\rho^>(\tau; \rho) = \int_0^\tau d\tau' I(\tau'; \rho) \rho^0(\tau - \tau'; |1\rangle) . \quad (66)$$

Thus one has

$$\rho(\tau) = e^{-M\tau} \rho e^{-M^\dagger \tau} + \int_0^\tau d\tau' I(\tau'; \rho) \rho^0(\tau - \tau'; |1\rangle) . \quad (67)$$

Differentiation gives

$$\dot{\rho}(\tau; \rho) = \dot{\rho}^0(\tau; \rho) + I(\tau; \rho) |1\rangle\langle 1| + \int_0^\tau d\tau' I(\tau'; \rho) \dot{\rho}^0(\tau - \tau'; |1\rangle) . \quad (68)$$

Taking the trace and using $\text{tr} \rho(\tau) \equiv 1$ gives

$$I(\tau; \rho) = A_3 \rho_{33}(\tau; \rho) \quad (69)$$

and thus Eq. (67) becomes in the present situation

$$\rho(\tau; \rho) = e^{-M\tau} \rho e^{-M^\dagger \tau} + \int_0^\tau d\tau' A_3 \rho_{33}(\tau'; \rho) \rho^0(\tau - \tau'; |1\rangle) . \quad (70)$$

From Eq. (64) one obtains $\dot{\rho}^0$, and inserting this into Eq. (68) gives

$$\dot{\rho}(\tau) = -\frac{i}{\hbar} [H_{\text{red}}^I \rho(\tau) - \rho(\tau) H_{\text{red}}^{I\dagger}] + A_3 \rho_{33}(\tau) |1\rangle\langle 1| . \quad (71)$$

This is a compact form of the Bloch equations used in Refs. [10, 11]. Conversely, from Eq. (71) one can immediately obtain the integral equation of Eq. (70).

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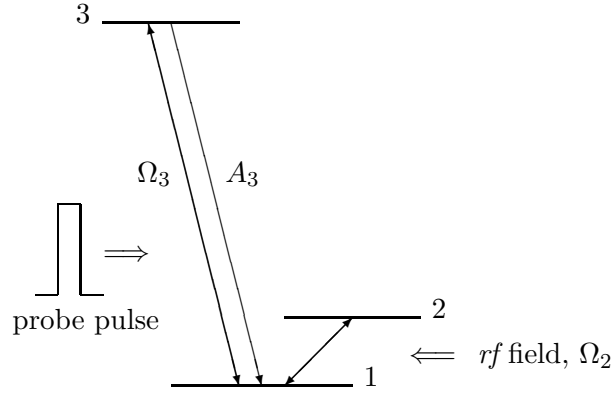


Figure 1: V system with (meta-)stable level 2 and auxiliary level 3 with Einstein coefficient A_3 . Ω_2 and Ω_3 are the Rabi frequencies of the *rf* field and the probe laser, respectively.

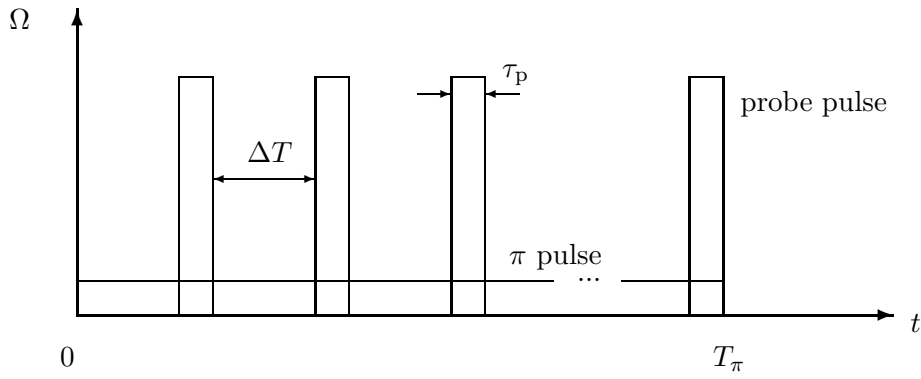


Figure 2: Probe pulses and π pulse